Unit - III

Lebesgue Integration

Riemann Integral:

Let f be a bounded real valued function defined on the interval $[a,b]$. Let $P: a = x_0 < x_1 < \cdots < x_n = b$ be a partition of $[a,b]$. The lower and upper Darboux sum are defined as,

$$L(f,P) = \sum_{i=1}^{n} m_i (x_i - x_{i-1}) \text{ where } m_i = \min f(x), \quad x_{i-1} \le x \le x_i$$

$$U(f,P) = \sum_{i=1}^{n} M_i (x_i - x_{i-1}) \text{ where } M_i = \max f(x), \quad x_{i-1} \le x \le x_i$$

$U(f,P) \ge L(f,P)$ for any partition P of $[a,b]$

If partition P' is finer than partition P (ie, $P' \supset P$) then $L(f,P') \ge L(f,P)$ and $U(f,P') \le U(f,P)$

combining, $U(f,P) \ge U(f,P') \ge L(f,P') \ge L(f,P)$

We define, $\overline{\int_a^b} f(x)\,dx = \inf U(f,P)$ where the infimum is taken over the partition of $[a,b]$ and then $\underline{\int_a^b} f(x)\,dx = \sup L(f,P)$ where the supremum is taken over the partition of $[a,b]$

$\overline{\int_a^b} f\,dx$ and $\underline{\int_a^b} f\,dx$ are called upper Riemann integral and lower Riemann integral of f over $[a,b]$ respectively

If $\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$, then this common value is called the integral of f over $[a,b]$ and is denote those integrals by,

$(R)\overline{\int_a^b} f$, $(R)\underline{\int_a^b} f$ and $(R)\int_a^b f$.

Defn:-

A real valued function ψ defined on $[a,b]$ is called a step function provided there is a partition $P=\{x_0, x_1, \cdots x_n\}$ of $[a,b]$ and numbers $c_1, c_2, \cdots, c_n$ such that for $1 \le i \le n$ $\psi(x) = c_i$ if $x_{i-1} \le x < x_i$

Remark :-

1) Suppose ψ is a step function on $[a,b]$.

$L(\psi, P) = \sum_{i=1}^{n} c_i (x_i - x_{i-1}) = U(\psi, P)$.

This shows that $(R)\underline{\int_a^b}\psi = (R)\overline{\int_a^b}\psi = (R)\int_a^b \psi$.

$\therefore$ ψ is Riemann integral.

$\therefore$ Every step function is Riemann integrable.

2) We can formulate the upper and lower Riemann integral of a bounded real valued function f on $[a,b]$ as follows:

$(R)\overline{\int_a^b} f = \inf\{(R)\int_a^b \psi \mid \psi$ a step function and $\psi \ge f$ on $[a,b]\}$

$(R)\underline{\int_a^b} f = \sup\{(R)\int_a^b \psi \mid \psi$ a step function and $\psi \le f$ on $[a,b]\}$

Example :

consider the Dirichlet's function $f : [0,1] \to R$ defined by $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$.

Let P be any partition of $[a,b]$

Then $L(f, P) = \sum_{i=1}^{n} m_i (x_i - x_{i-1}) = 0$ $\left[\because m_i = \min f(x) = 0,\ x_{i-1} \le x \le x_i\right]$

$U(f, P) = \sum_{i=1}^{n} M_i (x_i - x_{i-1}) = \sum_{i=1}^{n} (x_i - x_{i-1}) = x_n - x_0 = 1 - 0 = 1$

$(\because M_i = \max f(x) = 1,\ x_{i-1} \le x \le x_i)$

$\therefore (R)\underline{\int_0^1} f = 0$ and $(R)\overline{\int_0^1} f = 1$

$(R)\int_{-0}^{1} f \neq \int_{0}^{-1} f$

$\therefore f$ is not Riemann integrable.

Lebesgue integral of a bounded measurable function over a set of finite measure.

Defn:-

For a simple function φ defined in a set of finite measure E, we define the integral of φ over E by $\int_E \varphi = \sum_{i=1}^{n} a_i \, m(E_i)$,

where $E_i = \{x \in E / \varphi(x) = a_i\}$ and $\varphi = \sum_{i=1}^{n} a_i \chi_{E_i}$

Lemma:-

Let $\{E_i\}_{i=1}^{n}$ be a finite disjoint collection of measurable subset of a set of finite measure E. For $1 \leq i \leq n$, let a_i be a real number.

If $\phi = \sum_{i=1}^{n} a_i \chi_{E_i}$ on E, then $\int_E \phi = \sum_{i=1}^{n} a_i m(E_i)$

Pf:-

Since $\phi = \sum_{i=1}^{n} a_i \chi_{E_i}$ and E_i's are disjoint, ϕ can assume only a finite number of values (possibly n distinct values)

Since, $\chi_{E_i}(x) = \begin{cases} 1 & \text{if } x \in E_i \\ 0 & \text{if } x \notin E_i \end{cases}$

Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be the distinct values taken by ϕ.

For, $1 \leq j \leq m$, Let $A_j = \{x \in E \mid \phi(x) = \lambda_j\}$.

ϕ can be expressed as $\phi = \sum_{j=1}^{m} \alpha_j \chi_{A_j}$

This is a canonical representation $\int_E \phi = \sum_{j=1}^{m} \alpha_j \, m(A_j)$

Now,

for $1 \le j \le m$, let I_j be the set of indices i, in $\{1, 2, \ldots, n\}$ for which $a_i = \alpha_j$

Then $\{1, 2, 3, \ldots, n\} = \bigcup_{j=1}^{m} I_j$ and I_j's are disjoint and each I_j's are finite.

Then $A_j = \bigcup_{i \in I_j} E_i$

$\therefore m(A_j) = \sum_{i \in I_j} m(E_i)$ for $1 \le j \le m$

$$\therefore \sum_{i=1}^{n} a_i \, m(E_i) = \sum_{j=1}^{m} \Big[\sum_{j \in I_j} a_j \, m(E_j) \Big]$$

$$= \sum_{j=1}^{m} \Big[\alpha_j \sum_{j \in I_j} m(E_i) \Big]$$

$$= \sum_{j=1}^{m} \alpha_j \, m(A_j)$$

$$= \int_E \phi$$

proposition : (Linearity and monotonicity of integration)

Let ϕ and ψ be simple function defined on a set of measure E. Then for any α and β.

$$\int_E \alpha\phi + \beta\psi = \alpha \int_E \phi + \beta \int_E \psi$$

Moreover if $\phi \le \psi$ on E, then $\int_E \phi \le \int_E \psi$.

Pf :-

since both ϕ and ψ takes only a finite number of values on E, we may choose a finite disjoint collection $\{E_i\}_{i=1}^{n}$ of measurable subsets of E, the union of which is E, such that ϕ and ψ are constant on each on each E_i.

For each i, $1 \leq i \leq n$.

Let a_i and b_i respectively be the values taken by ϕ and ψ on E_i

By the proceeding lemma,

$$\int_E \phi = \sum_{i=1}^{n} a_i m(E_i) \text{ and } \int_E \psi = \sum_{i=1}^{n} b_i m(E_i)$$

However the simple function $\alpha\phi + \beta\psi$ takes the constant value $\alpha a_i + \beta b_i$ on E_i

Thus again by, the previous lemma,

$$\int_E (\alpha\phi + \beta\psi) = \sum_{i=1}^{n} (\alpha a_i + \beta b_i) m(E_i)$$

$$= \alpha \sum_{i=1}^{n} a_i m(E_i) + \beta \sum_{i=1}^{n} b_i m(E_i)$$

$$= \alpha \int_E \phi + \beta \int_E \psi.$$

To prove :- Monotonicity.

Assume $\phi \leq \psi$ on E.

Define $\eta = \psi - \phi$ on E.

By linearity $\int_E \psi - \int_E \phi = \int_E (\psi - \phi) = \int_E \eta \geq 0.$

($\because$ The non-negative simple function η has a non-negative)

Note :-

1) A step function is simple since it takes only a finite no of values and its domain $[a,b]$ is measurable.

2) The Riemann integral of a step function over a

closed and bounded interval agrees with the lebesgue integral.

3) Let f be a bounded real valued function defined on a set of finite measure E. We define the lower and upper Lebesgue integral respectively of f over E by,

$\sup\left\{\int_E \phi \,/\, \phi \text{ is simple and } \phi \leq f \text{ on } E\right\}$

$\inf\left\{\int_E \psi \,/\, \psi \text{ is simple and } \psi > f \text{ on } E\right\}$

is analogue to lower and upper Riemann integral.

Definition:

A bounded function f on a domain E of finite measure is said to be Lebesgue integrable over E provided its upper and lower lebesgue integrable over E are equal.

The common value of the upper and lower integrals is called the Lebesgue integral or simply the integral of f over E and is denoted by $\int_E f$

Theorem:

Let f be a bounded function defined on the closed bounded interval $[a, b]$. If f is Riemann integrable over $[a, b]$ then it is Lebesgue integrable over $[a, b]$ and the two integrals are equals.

Proof: Suppose f is Riemann integrable over $[a, b]$.

Let $I = [a, b]$.

$\sup L(f, P) = \inf U(f, P)$ where the sup & inf are taken over the partition of $[a, b]$.

In terms of step function,

$\sup\{(R)\int_I \phi \,/\, \phi$ is a step function, $\phi \le f\} = \inf\{R\int_I \psi \,/\, \psi$ is a step fn., $\psi \ge f\}$

W.K.T, each step function is a simple function —— ①.

Also, Riemann integral of step function agrees with the Lebesgue integral of the step function.

ie) $(R)\int_I \phi = \int_I \phi$ is the Lebesgue sense and

$(R)\int_I \psi = \int_I \psi$ is the Lebesgue sense —— ②

$\therefore \sup\{(R)\int_I \phi \,/\, \phi$ is a step function, $\phi \le f\}$.

$= \sup\{\int_I \phi \,/\, \phi$ is a simple function, $\phi \le f\}$ and

$\inf \{ (R) \int_I \psi / \psi$ is a step function ; $\phi \geq f \}$

$= \inf \{ \int_I \psi$ is a simple function, $\psi \geq f \}$ — (3)

(1) $\Rightarrow \sup \{ \int_I \phi / \phi$ is a simple function, $\phi \leq f \}$

$= \inf \{ \int_I \psi / \psi$ is a simple function, $\psi \geq f \}$.

$\therefore f$ is a lebesgue integral over $[a, b]$.

(2) $\Rightarrow (R) \int_I f = \int_I f$ (Lebesgue integral).

Example:

(1) Let E be the set of rational numbers in $[0, 1]$. Then E is a measurable set of measure zero.

For, consider the Dirichlet function.

$f : [0,1] \to R$. defined by $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational.} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

clearly, f is a restriction of $[0,1]$

To the characteristic function on E.

ie) $f = \chi_E$.

$\therefore \int_{[0,1]} f = \int_{[0,1]} \chi_E = 1 \cdot m(E) = 0$.

(2) Let $f : [0,1] \to R$ be defined by $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$

Find this Lebesgue integral of f over $[0,1]$.

Theorem:

Let f be a bounded measurable function on a set of finite measure E. Then f is integrable over E.

Proof

Let n be any natural number.

Let $\varepsilon = 1/n$.

By Simple approximation lemma, there exist two simple functions ϕ_n and ψ_n on E such that $\phi_n \le f \le \psi_n$ and $0 \le \psi_n - \phi_n \le \frac{1}{n} m(E)$.

By monotonicity and linearity of the lebesgue integrable of simple function,

$$0 \le \int_E \psi_n - \int_E \phi_n = \int_E (\psi_n - \phi_n) \le \frac{1}{n} m(E)$$

$$0 \le \int_E \psi_n - \int_E \phi_n \le \frac{1}{n} m(E).$$

Now, $\int_E \psi_n \ge \inf\{\int_E \psi : \psi \text{ is a simple function and } \psi \ge f\}$.

$-\int_E \phi_n \ge -\sup\{\int_E \phi : \phi \text{ is simple fn. } \& \ \phi \le f\}$

$\therefore \inf\{\int_E \psi : \psi \text{ is a simple function, } \psi \ge f\} -$
$\sup\{\int_E \phi : \phi \text{ is a simple function. } \phi \le f\} \le \int_E \psi_n - \int_E \phi_n$.

This holds for every natural number and $\le \frac{1}{n} m(E)$
$m(E)$ is finite

Also, upper lebesgue integral $\ge$ lower lebesgue integral.

$0 \le \inf\{\int_E \psi : \psi \text{ is a simple function } \psi \ge f\} \ge$
$\sup\{\int_E \phi : \phi \text{ is a simple function } \phi \le f\}$.

$\therefore 0 \le \inf\{\int_E \psi : \psi \text{ is a simple function } \psi \ge f\} -$
$\sup\{\int_E \phi : \phi \text{ is a simple function } \phi \le f\}$.
$\le \frac{1}{n} m(E) \to 0$ as $n \to \infty$.

$\inf\{\int_E \psi : \psi \text{ is a simple function } \psi \ge f\} =$
$\sup\{\int_E \phi : \phi \text{ is a simple function } \phi \le f\}$.

$\therefore f$ is a Lebesgue integral over E.

Theorem:

Linearity and Monotonicity

Let f and g be bounded measurable function on a set of finite measure E. Then for any α and β, $\int_E \alpha f + \beta g = \alpha \int_E f + \beta \int_E g$. Moreover if $f \le g$ on E, $\int_E f \le \int_E g$.

Proof: since α and β are real and f and g are bounded real valued fns. defined on E, $\alpha f + \beta g$ is bounded on E.

Also, f and g are measurable $\alpha f + \beta g$ is measurable on E

$\therefore \alpha f + \beta g$ is a bounded measurable fn. defined on the set E of finite measure.

$\therefore \alpha f + \beta g$ is Lebesgue integrable over E.

If ψ is a simple fn, then $\alpha\psi$ is also a simple fn and conversely, if $\alpha\psi$ is a simple fn, then ψ is a simple fn. provided $\alpha \neq 0$.

By the linearity of integration of simple fn,

$$\int_E \alpha f = \inf\{\int_E \psi : \psi \text{ is s f } \psi \ge \alpha f\}$$

$$= \inf\{\int_E \frac{\alpha}{\alpha}\psi : \frac{\psi}{\alpha} \text{ is a simple function, } \frac{\psi}{\alpha} \ge f\}.$$

$$= \alpha \inf\{\int_E (\frac{\psi}{\alpha}) / \frac{\psi}{\alpha} \text{ is a simple fn, } \frac{\psi}{\alpha} \ge f\}.$$

$$= \alpha \int_E f.$$

Let $\alpha < 0$.

Now, $$\int_E \alpha f = \inf\{\int_E \phi : \phi \text{ is a simple function, } \phi \ge \alpha f\}.$$

$$= \inf\{\int_E \phi(\frac{\alpha}{\alpha}) : \frac{\phi}{\alpha} \text{ is a simple function } \frac{\phi}{\alpha} \ge f\}$$

$$= \alpha \sup\{\int_E \phi/\alpha / \phi/\alpha \text{ is a simple function } \frac{\phi}{\alpha} \le f\}.$$

$$= \alpha \int f$$

Thus $\int_E \alpha f = \alpha \int_E f$ where $\alpha \neq 0$.

II The linearity of the integral. it is enought to prove

$$\int_E f + \int_E g = \int_E f+g$$

Let ψ_1 and ψ_2 be simple function such that $f \leq \psi_1$ & $g \leq \psi_2$ on E.

Then $\psi_1 + \psi_2$ is a simple function st $f+g \leq \psi_1 + \psi_2$.

$$\int_E f+g = \inf \left\{ \int_E \psi \,/\, \psi \text{ is a simple function } \psi \geq f+g \right\}$$

$$\leq \int_E \psi_1 + \psi_2 = \int_E \psi_1 + \int_E \psi_2$$

ie) $\int_E f + g \leq \int_E \psi_1 + \int_E \psi_2$ where $f \leq \psi_1$ & $g \leq \psi_2$.

Taking inf on both sides,

$$\int_E f+g \leq \inf \int_E \psi_1 + \inf \int_E \psi_2 = \int_E f + \int_E g$$

$$\int_E f+g \leq \int_E f + \int_E g \quad \text{— (1)}$$

II linearity. It is enough to prove $\int_E f+g \geq \int_E f + \int_E g$.

$$\therefore \int_E (f+g) = \sup \left\{ \int_E \phi \text{ is a simple function } \phi \leq f+g \right\}$$

$$\geq \int_E \phi_1 + \phi_2$$

$$= \int_E \phi_1 + \int_E \phi_2$$

ie) $\int_E f+g \geq \int_E \phi_1 + \int_E \phi_2$ where $f \geq \phi_1$ & $g \geq \phi_2$.

Taking sup on both sides,

$$\int_E f+g \geq \int_E f + \int_E g \quad \text{— (2)}$$

From (1) & (2). $\int_E f+g = \int_E f + \int_E g$

Suppose $g \geq f$ on E

Put $h = g - f$ on E.

Then $\int_E h = \int_E g-f = \int_E g - \int_E f$.

Since $g \geq f$ on E, $h \geq 0$ on E.

Let φ be the simple function defined by $\varphi = 0$ on E.

$\therefore \varphi \leq h$ on E.

Now, $\int_E h = \sup_{\varphi \leq h} \{ \int_E \varphi \mid \varphi$ is a simple function$\}$.

$$\int_E h \geq \int_E \varphi$$

$$\int_E h \geq 0.$$

$$\int_E h = \int_E g - \int_E f \geq 0.$$

$$\int_E g \geq \int_E f.$$

$$\int_E f \leq \int_E g.$$

Corrolary:

Let f be a bounded measurable function on a set of finite measure E. Then $\left|\int_E f\right| \le \int_E |f|$

Proof:

Since f is bounded measurable function on E.

$|f|$ is bounded measurable function on E and $-|f| \le f \le |f|$ on E

By the monotonicity of lebesgue interval

$$-\int_E |f| \le \int_E f \le \int_E |f|$$

$$\therefore \left|\int_E f\right| \le \int_E |f|.$$

Problem:

Let f be a bounded measurable function on a set of finite measure E. Then $\left|\int_E f\right| \le \int_E |f|$ for a measurable subset A of E show that $\int_A f = \int_E f \chi_A$

Proof:

Since f is bounded measurable function.

w.k.T $\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$

$\therefore \chi_A$ is measurable iff A is measurable

Given A is measurable subset of E, χ_A is a measurable function on E.

Also given that f is a measurable function on E.

$f\chi_A$ is measurable function on E.

Clearly χ_A is a measurable, bounded function on E.

$\therefore f\chi_A$ is bounded function on E.

$\therefore f \cdot \chi_A$ is a bounded measurable function on E.

Clearly $f/A = \chi_A \cdot f$.

Since $f \cdot \chi_A(x) = \begin{cases} f(x) & x \in A \\ 0 & x \in A \end{cases}$

Now, $\int_E f\chi_A = \int_{A \cup A^c} f\chi_A$.

$$= \int_A f \cdot \chi_A + \int_{A^c} f \cdot \chi_A = \int_A f + \int_{A^c} 0 = \int_A f.$$

Proposition:

Let f be a bounded measurable function on a finite measure on E. Suppose A and B are disjoint measurable subset of E Then $\int_{A \cup B} f = \int_A f + \int_B f$.

Proof:

Clearly $f \cdot \chi_A$ and $f \cdot \chi_B$ are bounded measurable function on E since A and B are disjoint.

$$f\chi_{A\cup B} = f\chi_A + f\chi_B$$

For Let $x \in E$, Then $f \cdot \chi_{A\cup B}(x) = \begin{cases} f(x) & \text{if } x \in A\cup B \\ 0 & \text{if } x \notin A\cup B. \end{cases}$

$$(f\cdot\chi_A + f\cdot\chi_B)(x) = f(x)\chi_A(x) + f(x)\chi_B(x).$$

$$= \begin{cases} f(x)+0 & \text{if } x\in A \text{ and } x\notin B \\ 0+f(x) & \text{if } x\notin A \text{ and } x\in B \\ 0+0 & \text{if } x\notin A \text{ and } x\notin B \end{cases}$$

$$= \begin{cases} f(x) & \text{if } x\in A\cup B. \\ 0 & \text{if } x\notin A\cup B. \end{cases}$$

$$= f\chi_{A\cup B}(x)$$

By the linearity of the above integration.

$$\int_{A\cup B} f = \int_A f + \int_B f.$$

Proposition

Let $\{f_n\}$ be a sequence of bounded measurable function on a set of finite measure E. If sequence $\{f_n\} \to f$ uniformly converges f on E. Then $\lim_{n\to\infty} \int_E f_n = \int_E f$.

Proof:

Given $\{f_n\}$ is a sequence of bounded measurable function
$\therefore \exists\ M \geq 0$ such that $|f_n| \leq M$ for all M.

Given seq $\{f_n\}$ converges f uniformly for every $\varepsilon > 0$ $\exists$ $n \in Z^+$ such that $|f_n(x) - f(x)| < \varepsilon$ $\forall$ $n \geq N$, $x \in E$.

ie) $|f_n - f| < \varepsilon$ $\forall$ $n \geq N$.

$|f - f_n| < \varepsilon$.

Now, $||f| - |f_n|| \leq |f - f_n| < \varepsilon$.

$\Rightarrow |f| - |f_n| < \varepsilon$.

$|f| < M + \varepsilon$.

f is bounded, since seq $\{f_n\} \to f$ uniformly on E. f is the pointwise limit of sequence $\{f_n\}$.

Thus f is a bounded measurable function on E and hence f is lebesgue integral.

Let $\varepsilon > 0$.

Since seq $\{f_n\} \to f$ uniformly $\exists$ an index N for which $|f - f_n| < \varepsilon / m(E)$ on E $\forall$ $n \geq N$.

Now, $\left|\int_E f_n - \int_E f\right| = \left|\int_E f_n - f\right| \leq \int_E \varepsilon/m(E) = \varepsilon/m(E) \cdot m(E)$.

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

The bounded convergence theorem:

Let $\{f_n\}$ be a sequence of measurable function on a set of finite measure E. Suppose seq $\{f_n\}$ is uniformly pointwise bounded on E. ie) there is a number $M \geq 0$ for which $|f_n| \leq M$ on E $\forall n$. If seq $\{f_n\} \to f$ pointwise on E Then $\lim_{n\to\infty} \int_E f_n = \int_E f$.

Proof:

Given $\{f_n\} \to f$ pointwise on E.

Since f_n are measurable function on E. f is measurable on E.

Also, Given that $\{f_n\}$ is uniformly pointwise bounded on E.

Clearly $\exists$ a number $M \geq 0$ for which $|f_n| \leq M$ on E $\forall n$.

$\therefore f$ is bounded.

Let $\varepsilon > 0$ be given.

Let A be any measurable subset of E.

Then $\int_E f_n - \int_E f = \int_A f_n - \int_A f + \int_{E\sim A} f_n - \int_{E\sim A} f$

$= \int_A f_n - f + \int_{E\sim A} f_n - \int_{E\sim A} f$.

$\left|\int_E f_n - \int_E f\right| = \left|\int_A f_n - f + \int_{E\sim A} f_n + \int_{E\sim A} -f\right|$

$\leq \int_A |f_n - f| + \int_{E\sim A} |f_n| + \int_{E\sim A} |f| \longrightarrow (*)$

Since $\{f_n\} \to f$ pointwise on E.

$\exists$ a closed set $F \subset E$ such that $\{f_n\} \to f$ uniformly on F and $m(E\sim F) < \varepsilon$.

Since a closed subset of a measurable set is measurable we can say that $\exists$ a measurable subset A of E such that $\{f_n\} \to f$ uniformly on A and $m(E\sim A) < \varepsilon/4M$.

Since $\{f_n\} \to f$ uniformly on A.

Given $\varepsilon > 0$ $\exists$ an index N such that $|f_n - f| < \frac{\varepsilon}{2M(A)}$ on A $\forall n \geq N$.

$* \Rightarrow \left|\int_E f_n - \int_E f\right| < \int_A \frac{\varepsilon}{2m(A)} + \int_{E\sim A} M + \int_{E\sim A} M$.

$= \frac{\varepsilon}{2m(A)} \times m(A) + M\, m(E\sim A) + M\, m(E\sim A)$.

$$= \varepsilon/2 + 2M + \varepsilon/4M$$

$$= \varepsilon/2 + \varepsilon/2$$

$$\left| \int_E f_n - \int_E f \right| < \varepsilon \quad \forall \; n \geq N$$

$$\therefore \lim_{n \to \infty} \int_E f_n = \int_E f$$

Lebesgue Integral of a measurable non-negative functions.

Definition: A measurable function f on E is said to vanish outside a set of finite measure provided there is a subset E_0 of E for which $m(E_0) < \infty$ and $f = 0$ on $E \sim E_0$

Definition:

The support of a measurable funct defined on E is. $\{x \in E \mid f(x) \neq 0\}$.

Note:

If f is bounded measurable function on E having a support of E_0 is finite measure, we define the integral of f ove E by, $\int_E f = \int_{E_0} f$ even if $m(E) = \infty$.

ie) If f is bounded measurable funct on E, vanishing outside a set of finite measure, then $\int_E f = \int_{E_0} f$, where $m(E_0) < \infty$ and $f = 0$ on $E \sim E_0$.

Definition: For a non-negative measurable function f, we define the integral of over E by. $\int f = \sup \{ \int_E h \mid h$ is a bounded measurable of finite support and $0 \leq h \leq f$

If $E_0 = \{x \in E / h(x) \neq 0\}$ with $m(E_0) < \infty$ then $\int_E h = \int_{E_0} h$.

Chebychev's inequality:

Let f be a non-negative measurable function on E. Then for any $\lambda > 0$,

$m(\{x \in E / f(x) \geq \lambda\}) \leq 1/\lambda \int_E f$.

<u>Proof</u>: Let $E_\lambda = \{x \in E / f(x) \geq \lambda\}$.

Case (i): Suppose $m(E_\lambda) = \infty$.

Let n be any natural numbers.

Define $E_{\lambda,n} = E_\lambda \cap [-n, n]$.

Clearly E_λ is an increasing sequence of subset of E for a field λ.

Let $\psi_n = \lambda \chi_{E_{\lambda,n}}$.

$$\psi_n(x) = \begin{cases} \lambda & \text{if } x \in E_{\lambda,n} \\ 0 & \text{if } x \notin E_{\lambda,n} \end{cases}$$

Since $E_{\lambda,n} \subset [-n, n]$, ψ_n is a bounded measurable function of finite support.

Since $E_{\lambda,n} \subset E_\lambda = \{x \in E / f(x) \geq \lambda\}$

$\therefore 0 \leq \psi_n \leq f$.

and $\int \psi_n = \lambda m(E_{\lambda,n}) \ \forall n$ —— (1)

Now, $E_\lambda = \bigcup_{n=1}^{\infty} E_{\lambda,n}$ and

$\{E_{\lambda,n}\}$ is an increasing sequence of subset of E_λ.

By continuity of measure we have

$$m\left(\bigcup_{n=1}^{\infty} E_{\lambda,n}\right) = \lim_{n\to\infty} m(E_{\lambda,n})$$

$$\Rightarrow m(E_\lambda) = \lim_{n\to\infty} m(E_{\lambda,n})$$

$$\lambda m(E_\lambda) = \lambda \lim_{n\to\infty} m(E_{\lambda,n})$$

$$= \lim_{n\to\infty} \lambda m(E_{\lambda,n})$$

$$= \lim_{n\to\infty} \int_E \psi_n.$$

$$\leq \int_E f$$

$$\therefore m(E_\lambda) \leq \frac{1}{\lambda} \int_E f.$$

$$\therefore m(\{x\in E \mid f(x) \geq \lambda\}) \leq \frac{1}{\lambda} \int_E f.$$

Case (ii): Suppose $m(E_\lambda) < \infty$.

Let $h = \lambda \chi_{E_\lambda}$.

$$h(x) = \begin{cases} \lambda & \text{if } x \in E_\lambda \\ 0 & \text{if } x \notin E_\lambda. \end{cases}$$

$\therefore$ h is bounded measurable function of finite support and $0 \leq h \leq f$.

$\therefore$ By definition of integral, $\int_E h \leq \int_E f$.

Now $\int_E h = \lambda m(E_\lambda)$.

$\textcircled{2} \Rightarrow \lambda m(E_\lambda) \leq \int_E f.$

$$m(E_\lambda) \leq \frac{1}{\lambda} \int_E f$$

ie) $m(\{x \in E \mid f(x) \geq \lambda\} \leq \frac{1}{\lambda} \int_E f.$

Proposition:

Let f be a non-negative measurable function on E. Then $\int_E f = 0$ iff $f = 0$ almost everywhere on E.

Proof: Let f be a non-negative measurable function on E.

Assume that $\int_E f = 0$

To prove : $f = 0$ almost everywhere on E.

ie to prove : $m\{x \in E / f(x) > 0\} = 0$

For each natural number n by Chebychev's inequality.

$$m(\{x \in E / f(x) \geq \frac{1}{n}\}) \leq n \int_E f = 0$$

ie) $m(\{x \in E / f(x) \geq \frac{1}{n}\}) \leq 0$

ie) $m(\{x \in E / f(x) \geq \frac{1}{n}\}) = 0 \quad \forall\, n = 1, 2, \ldots$

By the countable additivity of Lebesgu measure, $m(\{x \in E / f(x) > 0\}) = \sum_{n=1}^{\infty} m\{x \in E / f(x) \geq \frac{1}{n}$

$\therefore m(\{x \in E / f(x) > 0\}) = 0$

onversely,

Suppose $f = 0$ almost everywhere on E.

To prove : $\int_E f = 0$

Now, $\int_E f = \sup \{ \int_E h / h$ is a bounded measurable function having finite support and $0 \le h \le f$ on $E \}$.

Let ϕ be a simple function on E and h be a bounded measurable function on E having finite support and $0 \le \phi \le h \le f$ on E.

Since $f = 0$ almost everywhere on E, $\phi = 0$ almost everywhere on E.

Claim : $\int_E \phi = 0$

For, let $E_0 = \{ x \in E / \phi(x) \ne 0 \}$.

Then $m(E_0) = 0$.

Now, $\int_E \phi = \int_{E_0} \phi + \int_{E \sim E_0} \phi$

$= \int_{E_0} \phi + 0$

If $\phi = \sum_{i=1}^{n} c_i \chi_{E_i}$ with $\bigcup_{i=1}^{n} E_i = E_0$ then

$\int_{E_0} \phi = \sum_{i=1}^{n} c_i m(E_i)$.

Since $E_i \subseteq E_0 \ \forall i = 1, 2, \ldots n$ and $m(E_0) = 0$,

we have $m(E_i) = 0 \ \forall i = 1, 2, \ldots n$.

$\therefore \int_{E_0} \phi = 0$

Since this holds for all such simple functions ϕ, $\int_E h = 0$.

Since this holds for all set h,

i.e) h is a bounded measurable function having finite support and $0 \le h \le f$, $\int_E f = 0$.

Linearity and monotonically Integration:

Let f and g be non-negative measurable function on E for any $\alpha > 0$ and $\beta > 0$. $\int_E \alpha f + \beta g = \alpha \int_E f + \beta \int_E g$. Moreover if $f \le g$ on E then $\int_E f \le \int_E g$.

Proof: Let $\alpha > 0$.

Let h be a bounded measurable function on E with finite support and $0 \le h \le f$.

Since $\alpha > 0$, $0 \le h \le f$ iff $0 \le \alpha h \le \alpha f$.

Since h is bounded measurable function with finite support $\int_E \alpha h = \alpha \int_E h$.

[by linearity of integral and bounded measurable function with finite support]

$$\int_E \alpha f = \sup \left\{ \alpha \int_E h \,/\, 0 \le \alpha h \le \alpha f \right\}$$

$$= \alpha \sup \left\{ \int_E h \,/\, 0 \le h \le f \right\}$$

$$\int_E \alpha f = \alpha \int_E f.$$

It is sufficient to condition the case $\alpha = \beta = 1$. Let h and k be bounded measurable function. on E with finite support and $0 \le h \le f$, $0 \le k \le g$ on E.

$\therefore 0 \le h+k \le f+g$

By the linearity of integral of bounded measurable function with finite support.

$\int_E h + \int_E k = \int_E (h+k) \le \int_E (f+g)$

As h and k any among the bounded measurable function of finite support. The sup (lub) of LHS of the above inequality when $0 \le h \le f$ and $0 \le k \le g$ if $\int_E f + \int_E g$.

But $\int_E (f+g)$ is an upper bound of $\int_E h + \int_E k$

$$\therefore \int_E f + \int_E g \le \int_E f+g \quad \text{——— } ①$$

Now, $\int_E f+g = \sup \{ \int_E \ell / \ell$ is a bounded measurable function of finite support & $0 \le \ell \le f+g$

Define $h = \min(\ell, f)$.

$$\text{Then } h(x) = \begin{cases} \ell(x) & \text{if } \ell(x) \le f(x) \\ f(x) & \text{if } f(x) \le \ell(x). \end{cases}$$

Define $k = \ell - h$ clearly $k \ge 0$.

If $h(x) = \ell(x)$ then $k(x) = 0 \le g(x)$.

If $h(x) = f(x)$ then $k(x) = \ell(x) - f(x)$.

Also, $h(x) \le f(x)$.

Thus we have $0 \le h(x) \le f(x)$ and $0 \le k(x) \le$

and $\ell = h+k$.

Also from the definition of h and k, we inferthat h and k are bounded measurable function of finite support.

By the linearity of integration of bounded measurable function of finite support, $\int_E l = \int_E h + \int_E k$.

Since $0 \leq h \leq f$ and $0 \leq k \leq g$ by definition of integral of f and g we have.

$$\int_E h + \int_E k \leq \int_E f + \int_E g.$$

ie) $\int_E l \leq \int_E f + \int_E g$

By the definition of $\int_E (f+g)$

We have $\int_E f+g \leq \int_E f + \int_E g$ ——— ②

① and ②, $\int_E f+g = \int_E f + \int_E g$.

Let h be a bounded measurable function with finite support $0 \leq h \leq f$ on E

Suppose $f \leq g$ on E.

Then $0 \leq h \leq f \leq g$.

By the definition of $\int_E g$, $\int_E h \leq \int_E g$.

[$\because {}_E\int g = \sup \{ {}_E\int k \,/\, k$ is a bounded measurable function with finite support and $0 \le k \le g \}$]

i.e) ${}_E\int g$ is an upper bound for the collection

$\{ {}_E\int h \,/\, h$ is a bounded measurable function for $0 \le h \le f \}$.

But l.u.b of the collection ${}_E\int f$.

$\therefore {}_E\int f \le {}_E\int g$.

Factous lemma:

let $\{f_n\}$ be a sequence non-negative measure function on E. If $\{f_n\} \to f$ pointwise almost everywhere on E, then $\int_E f \leq \lim_{n\to\infty} \inf \int_E f_n$

Proof:

Given $\{f_n\}$ be a sequence of non-negative measurable function on E and $\{f_n\} \to f$ pointwise almost everywhere on E. Then f is a non-negative measurable function

we know that, E_0 is a subset of E with measure zero

$$\int_E f = \int_{E-E_0} f$$ for every non-negative measure function on E

without loss of generality we can assume that $\{f_n\} \to f$ pointwise on all of E

let h be any bounded measurable function finite support, for which $0 \leq h \leq f$ on E

Since h is bounded, $\exists M > 0$ such that $|h| \leq M$

Define $E_0 = \{x \in E \mid h(x) \neq 0\}$

then $m(E_0) < \infty$

let n be a natural number define h_n on E by $h_n = \min\{h, f_n\}$ on E

clearly, we observe that h_n is measurable

$0 \leq h_n \leq M$ on E_0 and $h_n = 0$ on $E \sim E_0$.

$\therefore \{h_n\}$ is uniformly bounded sequence of measurable function

$|f_n(x) - f(x)| < \epsilon \Rightarrow f(x) - \epsilon < f_n(x) < f(x) + \epsilon\}$

Also
support of $h_n(x) \leq$ support of h which has finite measure and by bounded convergence

$$\lim_{n\to\infty} \int_E h_n = \lim_{n\to\infty} \int_{E_0} h_n = \int_{E_0} h(x) = \int_E h(x)$$

Since $h_n \leq f_n$ by monotonicity

$$\lim_{n\to\infty} \int_E h_n \leq \lim_{n\to\infty} \int_E f_n$$

$$\text{(i.e.)} \int_E h \leq \liminf_{n\to\infty} \int_E f_n$$

$$\sup_{0\leq h\leq f} \int_E h = \liminf_{n\to\infty} \int_E f$$

$$\therefore \int_E f = \liminf_{n\to\infty} \int_E f_n$$

Thm
The montone convergence theorem.
let $\{f_n\}$ be a sequence of increasing non-negative measurable function on E - if $\{f_n\} \to f$ pointwise almost every where on E. then $\lim_{n\to\infty} \int_E f_n = \int_E f$

Proof.
Since $\{f_n\} \to f$ pointwise almost everyone by facto's lemma $\int_E f \leq \liminf_{n\to\infty} \int_E f_n$

Since $\{f_n\}$ is monotonically increasing and are non-negative $f_n \le f \forall n$

By monotonicity of lebesgue integral

$$\int_E f_n \le \int_E f$$

$$\lim_{n\to\infty} \sup \int_E f_n \le \int_E f$$

we know that $\lim_{n\to\infty} \inf \int_E f_n \le \lim_{n\to\infty} \sup \int_E f_n$

$$\therefore \int_E f_n \le \lim_{n\to\infty} \inf \int_E f_n \le \lim_{n\to\infty} \sup \int_E f_n \le \int_E f$$

$$\therefore \lim_{n\to\infty} \int_E f_n = \int_E f$$

$\Rightarrow$ limit exist and is equal to the common values.

<u>Corollary</u>:

Let $\{u_n\}$ be a sequence of non-negative measurable function on E & $f = \sum_{n=1}^{\infty} u_n$ pointwise almost everywhere on E then $\int_E f = \sum_{n=1}^{\infty} \int_E u_n$

<u>Proof</u>

Given $f = \sum_{n=1}^{\infty} u_n$ pointwise almost everywhere where $\{u_n\}$ is a sequence of non-negative measurable

let $f_n = \sum_{n=1}^{n} u_k$

clearly $\{f_n\}$ is monotonically increasing sequence of non-negative measurable function and $\{f_n\}$

(1)

$\therefore$ finite almost everywhere

By monotone convergence theorem

$$\lim_{n\to\infty}\int_E f_n = \int_E f \quad \text{—— (1)}$$

By linearity of lebesgue integral $\int_E f_n = \sum_{k=1}^{n}\int_E a_k$

$$\therefore \lim_{n\to\infty}\int_E f_n = \sum_{k=1}^{n}\int_E a_k$$

$$\sum_{n=1}^{\infty}\int_E U_n = \int_E f$$

Defn:

A non-negative measurable function on a measurable set E is said to be integrable over E provided $\int_E f < \infty$

Proposition 13

Let the non-negative function f be integrable over E then f is finite almost every where on E

Proof: Let n be any natural number

$\{x \; f(x) = \infty\} \subseteq \{x\in E \mid f(x) > n\}$

since f is integrable $\int_E f < \infty$

$\frac{1}{n}\int_E f \to 0$ as $n\to\infty$

$m\{x \in f(x) = \infty\} = 0$

f is finite almost everywhere on E

Beppo-Levi's Lemma

Let $\{f_n\}$ be increasing sequence of non-negative measurable function on E. If

Sequence of integrals $\{fn\}$ is bounded then $\{fn\}$ converges pointwise on E to a measurable function f that is finite almost everywhere on E and $\lim_{n\to\infty} \int_E fn = \int_E f < \infty$

Proof:
we know that monotonically increasing sequence in the extended real number system is convergent

$\therefore$ there exist a measurable function such that $\lim_{n\to\infty} fn(x) = f(x)$ Pointwise for all $x \in E$

By monotonic convergence thm,

$$\lim_{n\to\infty} \int_E fn = \int fn$$

Give then $\left|\int_E fn\right|$ is bounded

Since $\{fn\}$ is montonic increasing by monotic of integral of the sequence $\{\int_E fn\}$ is monoton increasing limit of $\int_E fn$ is infinite

(i) $\int_E f < \infty$

By the proposition f is finite everywhere.